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Introduction to Hyperfunctions and Their Integral Transforms

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Preface

This textbook is about generalized functions and some of their integral transforms in one variable. It is based on the approach introduced by the Japanese mathematician Mikio Sato. This is to be mentioned because the term hyperfunction that Sato has given to his generalization of the concept of function, is sometimes used today to denote generalized functions based on other approaches (distributions, Mikusinski's operators etc.). I have written this book, because I am delighted by the intuitive idea behind Sato's approach which uses the classical complex function theory to generalize the notion of function of a real variable. To my opinion, Sato's way to introduce the generalized concept of a function is less abstract than the one of Laurent Schwartz who defines his distributions as linear functionals on some space of test functions. On the other hand, I was quickly led to recognize that very few colleagues (mathematicians included) knew something about Satos's approach. Perhaps Sato and his school is not innocent for this state of affair. During several decades no elementary textbook addressing a wider audience was available (at least in English). Zealots delighted by the appealing intuitive idea of the approach have probably been rather rejected by the adopted style of exposition and the highly ambitious abstract mathematical concepts used in those few books. Fortunately, some years ago, I found Isac Imai's Book *Applied Hyperfunction Theory* which explains and applies Sato's hyperfunctions in a concrete, but nontrivial way, and thereby reveals their computational power. Without the help of Imai's book, I would have been definitely repelled too by the available sophisticated texts as perhaps many others before me. So, for the writing of my book I am indebted to Imai, mainly for the first chapter, parts of the second and entirely for the fifth chapter. The objective of my book is to offer an introduction to the theory of hyperfunctions and some of their integral transforms that should be readable to a wider audience (applied mathematicians, physicists, engineers) than to just some few specialists. The prerequisites are some basics notions on complex function theory of one variable and of the classical Laplace and Fourier transformation. Since I am no friend of theories for their own sake, I have inserted throughout the whole book some simple applications mainly to the theory of integral equations.

Chapter 1 is an elementary introduction to generalized functions by the hyperfunction approach of Sato. With a few basic facts on complex function theory the basic lines of a theory of generalized functions is presented that should be easy to read and easy to be understood. It shows the basic intuitive concept of a hyperfunction of one variable.

Chapter 2 discusses the analytic properties of hyperfunctions. Here, the specific methods of complex analysis come into bearing. Since a hyperfunction is defined by specifying a defining or generating function, we treat in some detail the question on how to construct a defining function such that the corresponding hyperfunction interprets the given ordinary function (problem of embedding an ordinary function in the set of hyperfunctions). We shall see that the answer to this problem is not always unique. A hyperfunction which interprets a given ordinary function on a specified interval is said to be a projection of this function to the interval. Given a hyperfunction on an interval (a', b'), the analogous problem of finding another hyperfunction that equals the specified one on a smaller interval $(a, b) \subset (a', b')$ and vanishes outside (a, b) is then treated. This leads to the notion of the so-called standard defining function, a term first defined for hyperfunctions with a compact support, then extended to hyperfunctions defined on an infinite interval. Imai extends the notion of a standard defining function in another way than Sato has done when the hyperfunction is not perfect, i.e. has a non-compact support. Because Imai's extension is found to be useful for the discussion of Hilbert transformation, I shall use the term "strong defining function" to avoid confusion of the two notions. An introduction to periodic hyperfunctions and their Fourier series then follows. The last theoretical part of this chapter discusses convolutions of hyperfunctions. Some informal remarks on integral equations conclude the chapter.

Chapter 3 treats the Laplace transform of hyperfunctions. It is somewhat the main axis of the part about integral transformations. While other texts about generalized functions often treat Fourier transforms in the first place and then used the established Fourier transformation to define the Laplace transformation, I will do it in the converse way. I introduce the Laplace transform of a hyperfunction by using a loop integral of the Hankel type over the defining function. Since simplicity of the presentation, together with many concrete examples without digging into finer mathematical points in the arguments, has been aimed, this more elementary approach in the main text is presented rather than Komatsu's theory of Laplace hyperfunctions. But since the Laplace transformation is a central subject of the book, I have presented an outline of Komatsu's approach in the Appendix B.

Chapter 4 is about Fourier transforms of hyperfunctions. Fourier transformation is without doubt the greatest beneficiary of the theory of generalized functions. Generally the Fourier transformation is treated in the first place in most approaches to generalized functions. From a mathematico-logical point of view this may be justified, however, from a computational and applied standpoint Laplace transformation is often more appropriate. I define the Fourier transform of a hyperfunction by using the already established Laplace transform. This approach has the advantage that the available extended tables of Laplace transforms can be used.

Chapter 5 treats Hilbert transforms of hyperfunctions. In this chapter I mainly follow Imai. The concept of a strong defining function plays an important role here because the existence of the Hilbert transform of a hyperfunction is intimately connected to the existence of a strong defining function. A section on analytic signals can also be found there.

For the last two chapters about Mellin and Hankel transformations I had to be entirely self-supporting. While there is an abundant literature about classical Mellin and Hankel transformation of ordinary functions, I could not find anything at least in English, German or French on Mellin or Hankel transformation of hyperfunctions. A. H. Zemanian [40] treats the Mellin and Hankel transformation of generalized functions based on Laurent Schwartz's theory of distributions. I have finally succeeded to convey the Mellin transformation to hyperfunctions by taking advantage that the Mellin transformation is in some sense a reformulation of the two-sided Laplace transformation. A simple change of variables allows passing from the Laplace to the Mellin transformation, and vice versa. Because the Laplace transformation of hyperfunction has been firmly established in Chapter 3, it was finally straightforward to establish the Mellin transformation of hyperfunctions by exploiting this connection to the hilt.

It was harder to define the Hankel transform of a hyperfunction. I have eventually found a way by working on the line of Mac Robert's proof of the classical Hankel transformation which uses the so-called Lommel integrals of Bessel functions. The established theory about the generalized Hankel transformation then finally works for hyperfunctions of slow growth.

Throughout the whole book a particular function and its hyperfunction counterpart comes up again and again: the Heaviside function $Y(x)$ and the unit-step hyperfunction $u(x)$. For didactic reasons, and after some hesitations, I have made a distinction between them. This may be pedantic and I agree that it is not absolutely necessary because there is no great danger of confusion by the use of a unified notation for the two. Thus, the reader should feel free to replace everywhere $Y(x)$ by $u(x)$. Systematically, all contours in the complex plane will be positively directed unless noted to the contrary. This will produce minus signs before integrals where some readers will not expect them, perhaps.

The main tool used in this book consists of contour integration in the complex plane. You will find numerous integrals taken on a closed contour or on an infinite loop. We liberally interchange in many places the order of integration in multiple integrals or the integral and an infinite series. In order to keep the flow of the arguments fluid, I do not generally justify these steps in detail. However, at the disposition for readers interested in such technical details, I have collected in the Appendix A3 the principal theorems generally used to justify such interchanges of limit operations.

Lastly, a few remarks about what you cannot find in this book and about what I do not have any pretensions. The theory goes not very deep. No sheafs and other sophisticated concepts are mentioned. The intended message is rather conveyed through many concrete examples. Also, and this is certainly a shortcoming, no hyperfunctions of several variables are treated.

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