Package Commands V. 7.0

In[1]:=
```
<< LaplaceAndzTransforms`LZSession`;
InitLZPackage[];
```

What is new in V.7.0 and 6.0?

LaplaceImageExpression, LaplaceImageSystem

Given any integro-differential expression obtain its Laplace transform:

```
LaplaceImageExpression[
 B Convolution[x''[u], Cos[u], u][t] + 4 Convolution[x[u], Sin[u], u][t] +
  x''[t] - x[t] + 3 y'[t] + 3 Cos[t] - a t, {x, y}, {X, Y}, t, s]
```

$$-\frac{a}{s^2}+\frac{3\,s}{1+s^2}-X+s^2\,X+\frac{4\,X}{1+s^2}-s\,x[0]+3\,(s\,Y-y[0])+\frac{B\,s\,\left(s^2\,X-s\,x[0]-x'[0]\right)}{1+s^2}-x'[0]$$

Given any integro-differential system and obtain the Laplace transform of its solution:

```
LaplaceImageSystem[{x''[t] - x[t] == 3 y'[t] + Convolution[u, y[u], u][t] - a t,
  y''[t] - x'[t] == t}, {x, y}, {X, Y}, t, s]
```

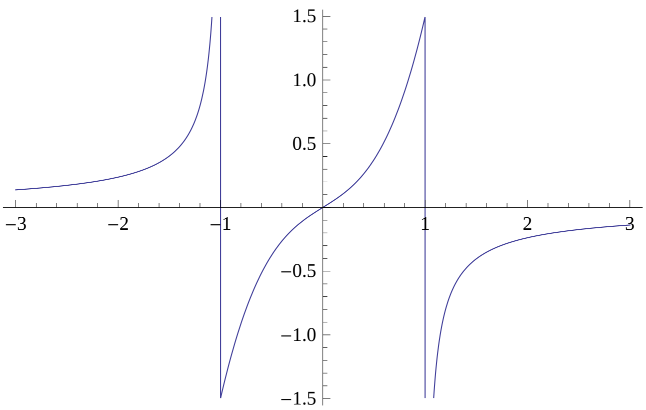
$$\Big\{X\to\frac{1}{s^3\left(-1-4\,s^3+s^5\right)}\left(1+3\,s^3-a\,s^4-s^2\,x[0]-3\,s^5\,x[0]+s^7\,x[0]+s^3\,y[0]+s^6\,x'[0]+s^2\,y'[0]+3\,s^5\,y'[0]\right),$$
$$Y\to\frac{-1-a\,s+s^2+s^2\,x[0]-4\,s^3\,y[0]+s^5\,y[0]+s^3\,x'[0]-s^2\,y'[0]+s^4\,y'[0]}{s\left(-1-4\,s^3+s^5\right)}\Big\}$$

Hilbert transforms of functions and generalized functions

```
g = HilbertTransform[Heaviside[t, -1, 1] t^4 / Sqrt[1 - t^2], t, y]
```

$$\frac{\frac{\sqrt{\pi}\,y}{2}+\sqrt{\pi}\,y^3}{\sqrt{\pi}}-\frac{y^4\,\mathrm{sgn}[y,-1,1]}{\sqrt{-1+y^2}}$$

```
Plot[ReleaseHold[g], {y, -3, 3}]
```



Extensions of the Laplace transformation to generalized Functions, as for example

```
LaplaceTransform[Heaviside[t, 0, b] Pf[1 / (t - b)^n], t, s]
```

$$(-1)^n\,b^{1-n}\,e^{-b\,s}\,\mathrm{ExpIntegralEinS}[n,\,b\,s]$$

```
DefInteger[n]; DefPositive[n];
InverseLaplaceTransform[s^n Log[s], s, t]
```

$(-1)^{1+n}\, n!\, \mathrm{Pf}\left[t^{-1-n}\right] + (-\mathrm{EulerGamma} + \mathrm{HarmonicNumber}[n])\, \mathrm{DiracDelta}[t]^{(n)}$

Computes all original functions corresponding to a given rational two-sided Laplace image functions, as for example

$$F(s) = \frac{s^2 - 3s + 5}{(s+1)^2 (s-1) (s-1-i)^2 (s-1+i)^2}$$

For factors in the denominator containing pairs of conjugate complex roots only one has to be specified.

```
InverseTwoSidedLaplaceTransform[s^2 - 3 s  +5, (s + 1)^2 (s - 1) (s - 1 - I)^2, s, t]
```

$$\Big\{\Big\{-\infty < \mathrm{Re}[s] < -1,\; -\left(-\frac{139\, e^{-t}}{500} + \frac{3\, e^{t}}{4} - \frac{9\, e^{-t}\, t}{50} + 2\, e^{t}\left(-\frac{59\,\mathrm{Cos}[t]}{250} - \frac{149\,\mathrm{Sin}[t]}{500}\right) + 2\, e^{t}\, t\left(\frac{11\,\mathrm{Cos}[t]}{100} - \frac{\mathrm{Sin}[t]}{50}\right)\right) \mathrm{UnitStep}[-t]\Big\},$$

$$\Big\{-1 < \mathrm{Re}[s] < 1,\; -\left(\frac{3\, e^{t}}{4} + 2\, e^{t}\left(-\frac{59\,\mathrm{Cos}[t]}{250} - \frac{149\,\mathrm{Sin}[t]}{500}\right) + 2\, e^{t}\, t\left(\frac{11\,\mathrm{Cos}[t]}{100} - \frac{\mathrm{Sin}[t]}{50}\right)\right) \mathrm{UnitStep}[-t] + \left(-\frac{139\, e^{-t}}{500} - \frac{9\, e^{-t}\, t}{50}\right) \mathrm{UnitStep}[t]\Big\},$$

$$\Big\{1 < \mathrm{Re}[s] < \infty,\; \left(-\frac{139\, e^{-t}}{500} + \frac{3\, e^{t}}{4} - \frac{9\, e^{-t}\, t}{50} + 2\, e^{t}\left(-\frac{59\,\mathrm{Cos}[t]}{250} - \frac{149\,\mathrm{Sin}[t]}{500}\right) + 2\, e^{t}\, t\left(\frac{11\,\mathrm{Cos}[t]}{100} - \frac{\mathrm{Sin}[t]}{50}\right)\right) \mathrm{UnitStep}[t]\Big\}\Big\}$$

Fractional derivatives and integrals, as for example

```
FractionalDerivative[1 / 2, t^3 / 2, t]
```

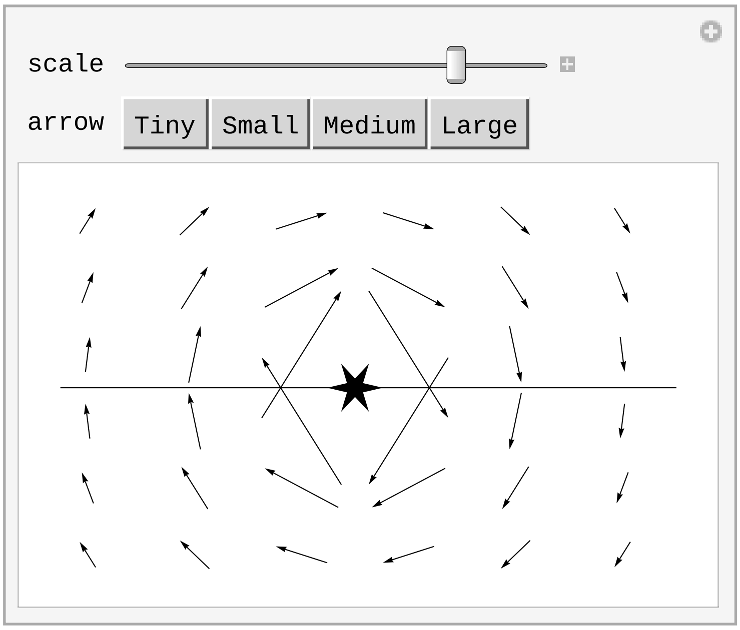
$$\frac{8\, t^{5/2}}{5\sqrt{\pi}}$$

```
FractionalIntegral[1 / 2, t^3 / 2, t]
```

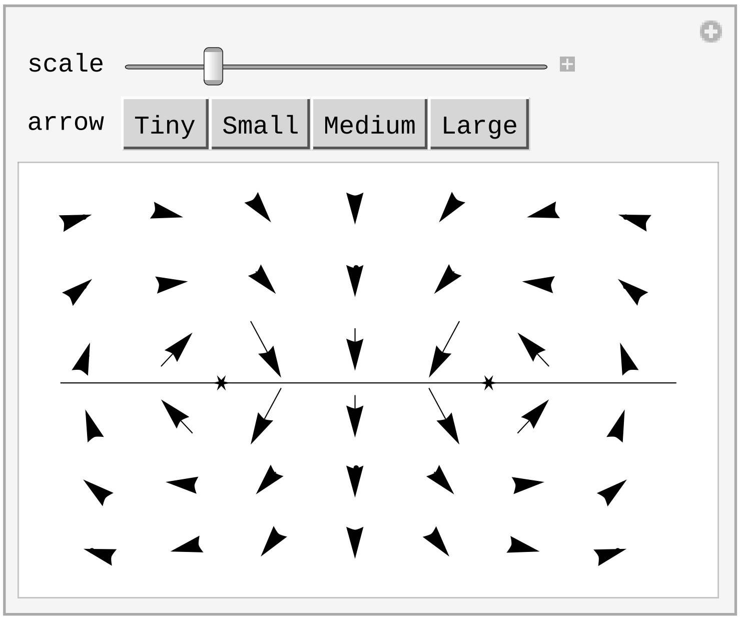
$$\frac{16\, t^{7/2}}{35\sqrt{\pi}}$$

Hyperfunctions and their visualisations, as for example

```
Manipulate[Visualize2D[Hyperfunction[-1 / (2 Pi I z), z], {-1, 1, 5}, {0.5, 2},
  scale, arrow, {0}], {scale, 0.1, 1}, {arrow, {Tiny, Small, Medium, Large}}]
```




```
Manipulate[Visualize2D[Hyperfunction[-1 / (2 Pi I (z + 0.5)) + 1 / (2 Pi I (z - 0.5)), z],
  {-1, 1, 6}, {0.5, 2}, scale, arrow, {-0.5, 0.5}],
 {scale, 0.1, 1}, {arrow, {Tiny, Small, Medium, Large}}]
```



- **Nonlinear Autonomous Oscillations**

Consider for example Duffin's equation

$y'' + \omega^2 y(t) = \epsilon y^3(t)$

where ϵ is a small parameter. The nonlinear problem can be considered being a perturbation of the linear one. Putting

$y(t) = y_0(t) + \epsilon\, y_1(t) + \epsilon^2\, y_2(t) + \ldots$
you might think of solving the the above problem by substituting the expression of $y(t)$ into the differential equation and solving sucessively. This will do no good since so-called secular terms will arise (powers of t), while the expected solution is a periodic one.

```
StraightforwardExpansion[(p^2 + ω^2)[y] == ϵ y^3,
 {y[0] → a Cos[ϕ], y'[0] → -a Sin[ϕ]}, p, y[t], t, 1]
```

$$\frac{-a\,\text{Cos}[\phi - t\,\omega] + a\,\omega\,\text{Cos}[\phi - t\,\omega] + a\,\text{Cos}[\phi + t\,\omega] + a\,\omega\,\text{Cos}[\phi + t\,\omega]}{2\,\omega} +$$

$$\frac{1}{256\,\omega^5}\,\epsilon\,\Big(-3\,a^3\,\text{Cos}[\phi - 3\,t\,\omega] + 3\,a^3\,\omega\,\text{Cos}[\phi - 3\,t\,\omega] + 3\,a^3\,\omega^2\,\text{Cos}[\phi - 3\,t\,\omega] - 3\,a^3\,\omega^3\,\text{Cos}[\phi - 3\,t\,\omega] +$$
$$a^3\,\text{Cos}[3\,\phi - 3\,t\,\omega] - 3\,a^3\,\omega\,\text{Cos}[3\,\phi - 3\,t\,\omega] + 3\,a^3\,\omega^2\,\text{Cos}[3\,\phi - 3\,t\,\omega] - a^3\,\omega^3\,\text{Cos}[3\,\phi - 3\,t\,\omega] -$$
$$27\,a^3\,\text{Cos}[\phi - t\,\omega] - 3\,a^3\,\omega\,\text{Cos}[\phi - t\,\omega] - 21\,a^3\,\omega^2\,\text{Cos}[\phi - t\,\omega] + 3\,a^3\,\omega^3\,\text{Cos}[\phi - t\,\omega] +$$
$$9\,a^3\,\text{Cos}[3\,\phi - t\,\omega] + 3\,a^3\,\omega\,\text{Cos}[3\,\phi - t\,\omega] - 21\,a^3\,\omega^2\,\text{Cos}[3\,\phi - t\,\omega] + a^3\,\omega^3\,\text{Cos}[3\,\phi - t\,\omega] +$$
$$27\,a^3\,\text{Cos}[\phi + t\,\omega] - 3\,a^3\,\omega\,\text{Cos}[\phi + t\,\omega] + 21\,a^3\,\omega^2\,\text{Cos}[\phi + t\,\omega] + 3\,a^3\,\omega^3\,\text{Cos}[\phi + t\,\omega] -$$
$$9\,a^3\,\text{Cos}[3\,\phi + t\,\omega] + 3\,a^3\,\omega\,\text{Cos}[3\,\phi + t\,\omega] + 21\,a^3\,\omega^2\,\text{Cos}[3\,\phi + t\,\omega] + a^3\,\omega^3\,\text{Cos}[3\,\phi + t\,\omega] +$$
$$3\,a^3\,\text{Cos}[\phi + 3\,t\,\omega] + 3\,a^3\,\omega\,\text{Cos}[\phi + 3\,t\,\omega] - 3\,a^3\,\omega^2\,\text{Cos}[\phi + 3\,t\,\omega] - 3\,a^3\,\omega^3\,\text{Cos}[\phi + 3\,t\,\omega] -$$
$$a^3\,\text{Cos}[3\,\phi + 3\,t\,\omega] - 3\,a^3\,\omega\,\text{Cos}[3\,\phi + 3\,t\,\omega] - 3\,a^3\,\omega^2\,\text{Cos}[3\,\phi + 3\,t\,\omega] - a^3\,\omega^3\,\text{Cos}[3\,\phi + 3\,t\,\omega] +$$
$$36\,a^3\,t\,\omega\,\text{Sin}[\phi - t\,\omega] - 12\,a^3\,t\,\omega^2\,\text{Sin}[\phi - t\,\omega] + 12\,a^3\,t\,\omega^3\,\text{Sin}[\phi - t\,\omega] - 36\,a^3\,t\,\omega^4\,\text{Sin}[\phi - t\,\omega] -$$
$$12\,a^3\,t\,\omega\,\text{Sin}[3\,\phi - t\,\omega] + 12\,a^3\,t\,\omega^2\,\text{Sin}[3\,\phi - t\,\omega] + 12\,a^3\,t\,\omega^3\,\text{Sin}[3\,\phi - t\,\omega] -$$
$$12\,a^3\,t\,\omega^4\,\text{Sin}[3\,\phi - t\,\omega] + 36\,a^3\,t\,\omega\,\text{Sin}[\phi + t\,\omega] + 12\,a^3\,t\,\omega^2\,\text{Sin}[\phi + t\,\omega] +$$
$$12\,a^3\,t\,\omega^3\,\text{Sin}[\phi + t\,\omega] + 36\,a^3\,t\,\omega^4\,\text{Sin}[\phi + t\,\omega] - 12\,a^3\,t\,\omega\,\text{Sin}[3\,\phi + t\,\omega] -$$
$$12\,a^3\,t\,\omega^2\,\text{Sin}[3\,\phi + t\,\omega] + 12\,a^3\,t\,\omega^3\,\text{Sin}[3\,\phi + t\,\omega] + 12\,a^3\,t\,\omega^4\,\text{Sin}[3\,\phi + t\,\omega]\Big)$$

There are several methods proposed in the past to overcome these difficulties: Lindstedt's Method, the method of multiple scales and the method of the Russian School Krylov-Bogolioubov-Mitropolski (KBM). Programed is a combination of the multiple scales and KBM.

In order to obtain a first order approximation:

```
AutonomousOscillation1st[y^3, y, y', ω, ϵ, a, ψ, t]
```

$$\left\{a'[t] == 0,\ \psi'[t] == \omega,\ y \to a\,\text{Cos}[\psi] - \frac{a^3\,\epsilon\,\text{Cos}[3\,\psi]}{32\,\omega^2}\right\}$$

In order to obtain a second order approximation:

```
AutonomousOscillation2nd[y^3, y, y', ω, ϵ, a, ψ, t]
```

$$\left\{a'[t] == 0,\ \psi'[t] == -\frac{15\,a^4\,\epsilon^2}{256\,\omega^3} - \frac{3\,a^2\,\epsilon}{8\,\omega} + \omega,\right.$$
$$\left. y \to a\,\text{Cos}[\psi] - \frac{a^3\,\epsilon\,\text{Cos}[3\,\psi]}{32\,\omega^2} + \frac{a^5\,\epsilon^2\,(-21\,\text{Cos}[3\,\psi] + \text{Cos}[5\,\psi])}{1024\,\omega^4}\right\}$$

For the van der Pol Oscillator:

$$y'' + \omega^2\,y(t) = \epsilon\left[1 - y^2(t)\right] y'(t)$$

```
AutonomousOscillation1st[(1 - y^2) dy, y, dy, ω, ϵ, a, ψ, t]
```

$$\left\{a'[t] == -\frac{\epsilon\,\left(-4\,a\,\omega + a^3\,\omega\right)}{8\,\omega^2},\ \psi'[t] == \omega,\ y \to a\,\text{Cos}[\psi] - \frac{a^3\,\epsilon\,\text{Sin}[3\,\psi]}{32\,\omega}\right\}$$

```
AutonomousOscillation2nd[(1 - y^2) dy, y, dy, ω, ϵ, a, ψ, t]
```

$$\left\{a'[t] == \frac{\epsilon\left(-4\,a\,\omega + a^3\,\omega\right)}{8\,\omega},\ \psi'[t] == \epsilon^2\left(-\frac{3}{8\,\omega} + \frac{3\,a^2}{8\,\omega} - \frac{19\,a^4}{256\,\omega}\right) + \omega,\right.$$
$$\left. y \to a\,\mathrm{Cos}[\psi] - \frac{a^3\,\epsilon^2\left(3\left(8 + a^2\right)\mathrm{Cos}[3\,\psi] + 5\,a^2\,\mathrm{Cos}[5\,\psi]\right)}{3072\,\omega^2} - \frac{a^3\,\epsilon\,\mathrm{Sin}[3\,\psi]}{32\,\omega}\right\}$$

List of all Commands in V. 7.0

▪ Loding the Package

```
<< LaplaceAndzTransforms`LZSession`;
InitLZPackage[]
```

or,

```
<< LaplaceAndzTransforms`LZSession`;
InitLZPackage[IncludeBuiltInCode → False]
```

▪ Laplace Transforms

```
LaplaceTransform[t^5 Cos[a t],t,s]

LaplaceTransform[Cos[c t]//ScaleChange[c,t],t,s]

LaplaceTransform[Cos[c t]//LeftShiftRule[c,t],t,s]//Simplify

LaplaceTransform[BesselJ[4,t]//IntegrationTo[t],t,s]

LaplaceTransform[Sin[t]//DivisionBy[t],t,s]

LaplaceImageExpression[
 Convolution[x''[u], Cos[u], u][t] + 4 Convolution[x[u], Sin[u], u][t] +
  x''[t] - x[t] + 3 y'[t] + 3 Cos[t] - a t, {x, y}, {X, Y}, t, s]

DiscreteLaplaceTransform[k Cos[k], k, T, s]

f = PeriodicFunction[#/(2 Pi)&,2 Pi];
LaplaceTransform[f[t], t, s]

f = PiecewiseFunction[{{0,1,0&},{1,2,(#-1)&},{2,4,1&},{4,5,
(1-(#-4))&}, {7,8,2&}}];
LaplaceTransform[f[t], t, s]

f = StepFunction[#^2&,0.5];
LaplaceTransform[f[t], t, s]

DiracDeltaRow[(# Exp[-#])&,T];
LaplaceTransform[%[t], t, s]

LaplaceTransform[Derivative[2][DiracDelta[t - a]], t, s]

LaplaceTransform[Summation[Derivative[2][DiracDelta[t - k T]], {k, 0, Infinity}], t, s]

LaplaceTransform[Pf[1 / t^4], t, s]
```

```
DefPositive[n]; DefInteger[n];
LaplaceTransform[Pf[1 / t^n], t, s]

LaplaceTransform[Pf[t^(-9 / 8)], t, s]

LaplaceTransform[Heaviside[t, 0, b] Pf[1 / (t - b)^n], t, s]

LaplaceTransform[Pf[BesselJ[0, t] / t], t, s]

LaplaceTransform[Pf[Cosh[ a t] / t], t, s]

TwoSidedLaplaceTransform[UnitStep[t-a] Cos[b t],t,s]//Simplify
```

- **Inverse Laplace Transforms**

```
InverseLaplaceTransform[Sqrt[s] / (s - a), s, t]

InverseTwoSidedLaplaceTransform[{1 / (s + 5)^4, Interval[{-Infinity, -5}]}, s, t]

InverseLaplaceTransform[(s^2+s-1)/(a s^2 + b s + c)^2,s,t,
Assumptions->b^2-4 a c ==0]

InverseLaplaceTransform[1/(s(s^2 + 2 s + 17)(1-Exp[-2 s])),s,t,
Transient->True]

InverseLaplaceTransform[s^n Log[s], s, t]

InverseLaplaceTransform[ExpIntegralE[1, b s], s, t]

DefInteger[n]; DefPositive[n];
InverseLaplaceTransform[s^n Log[s], s, t]

InverseLaplaceTransform[((s+1)/(s^2+a^2))//LeftShiftRule[c,s],s,t]

InverseLaplaceTransform[(s+1)/(s^2+a^2)//RightShiftRule[c,s],s,t]

InverseLaplaceTransform[(s/(s^2 + 2 s + 5)^2)//ImageShift[c,s],s,t]

InverseLaplaceTransform[1/Sqrt[1+s^2]//DivisionBy[s],s,t]

InverseLaplaceTransform[Exp[-a s + b]/Sqrt[s^2+a^2],s,t]
/. ImageRules

InverseLaplaceTransform[F[s]//Subst[s,Sqrt[s]],s,t]

InverseLaplaceTransform[(F[s]//Subst[s,Log[s]])/s,s,t]

InverseLaplaceTransform[Resolvent[s, {{1, -1}, {3, -4}}], s, t] /. t → t - t0

ExpansionTheorem[s + 1, s^3 (s^4 - 1) (s^2 + 2 s + 5), {1, -1}, {I, -1 + 2 I}, s, t]

ExpansionTheorem[Sinh[x s], s^2 Cosh[a s] (s - b),
 {b}, Hold[Table[I Pi (1 / 2 + n) / a , {n, 0, Infinity}]], s, t]

ResidueLaplace[(s^2 + s + d)/(s^2 + ω ^2)^3,{ω ⅈ ,-ω ⅈ },s,t]

InitialValues[(a s + b)/(c s^2 + d s +1)^2,s,6]

InverseTwoSidedLaplaceTransform[{1 / (s - a)^4, Interval[{a, Infinity}]}, s, t]

InverseTwoSidedLaplaceTransform[s^2 - 3 s 1 + 5,
 {{1 + I, 2}, {-2, 3}, {3 - 5 I, 1}, {2, 2}}, s, t]

InverseTwoSidedLaplaceTransform[-s + s^2, {{-1, 2}, {I, 1}, {2, 1}}, s, t]

InverseTwoSidedLaplaceTransform[{1/(s^2-a^2),Interval[{-a,a}]},
s,t]
```

Hilbert Transforms

```
HilbertTransform[x Exp[- Abs[x]], x, y]

HilbertTransform[DiracDelta[t - 1], t, y]

HilbertTransform[DiracComb[x, 1], x, y]

HilbertTransform[Pf[Cot[Pi x / T]] , x, y]

HilbertTransform[TriangleFunction[x, a], x, y]

HilbertTransform[x Sign[x] Heaviside[x, -5, 5], x, y]

HilbertTransform[f[x] // ScaleChange[c, x], x, y]
```

Functions, Generalized Functions and Sequences

```
FractionalDerivative[1 / 2, t^3 / 2, t]

FractionalIntegral[1 / 2, t^3 / 2, t]

Visualize[Hyperfunction[-1 / (2 Pi I) / z, z], -1, 1, "Dirac Impulse", 3, 0.1]

DiracComb[x, T]

DiracDeltaRow[(# Exp[-#]) &, T]

Heaviside[x, a, b]

sgn[x, a, b]

DiracDelta[x - b]

UnitStep[x - b]

PeriodicFunction[#/(2 Pi)&,2 Pi]

f = PeriodicEvenFunction[#^2&,0.5];
Plot[f[t], {t, -2, 2}]

PeriodicOddFunction[#^2&,0.5]

Periodic3dsFunction[#^2&,0.25]

f = PiecewiseFunction[{{0,1,0&},{1,2,(#-1)&},{2,4,1&},
{4,5,(1-(#-4))&}, {7,8,2&}}];
Plot[f[t], {t, -1, 10}]

StepFunction[#^2&,0.5]

SpecialPartialFraction[(s^7 + a)/((s^2 + 2 s + 17)^2 (s+b)),s]

SpecialPartialFraction[z^2/((z^2 + 2 z + 17)(z-3)^2),z,
Type -> ZTransformation]

SpecialPartialFraction[1/((s^2 + 2 s + 2)^2 (s-a)2 (s-2)^2),s,
Terms -> DropLast[{-1-I,-1+I,a}]]

PartialFraction[(s^6+s+1)/((s^2 + 2 s + 26)^2 (s+a)^2),s]

GeneralPartialFraction[(s^6+s+1)/((s^2 + 2 s + 26)^2 (s+3)^2),s,c,a]

GeneralPartialFraction[z^2/(z^2 + 2 z + 17),z,c,a,
Type -> ZTransformation]
```

```
GeneralPartialFraction[s/((s-a)^2 (s^2 + ω ^2)^2),s,c,a,
Terms -> DropLast[{ⅈ ω , -ⅈ ω }]]

IncompletePartialFraction[(s^2 + s + 1)/((3 s^2 + 2 s + 5)^3 *
(s+a)^2),h,k,s]

SimplifyRImageFunction[(s+a)^2/(s^2 + a s^2 - b s + c s - d s + e),s]

NormalizeRImageFunction[(a0 + a1 s + a2 s^2)/(b0 + b1 s + b2 s^2 + b3 s^3),s]

Convolution[Cos[u],Exp[u],u][t] /. EvaluateConvolution

Convolution[Cos[u],Exp[u],u][t] /. ComputeConvolution

DiscreteComposition[Exp[-x],Sin[x],x,T][t,τ ]
/. ComputeDiscreteComposition

FrequencyResponse[s/(s^2 + 2 s + 5),s,a Cos[ω t + φ ],t]

FrequencyResponse[s/(s^2 + 2 s + 5),s,a Cos[ω t + φ ],t]

BodePlot[s/(s^2 + 0.2 s + 1),{s,0.5,3}]

ContinuousArgPlot[Arg[ω I/(-ω ^2+ 0.2 I ω + 1)],{ω ,0.8,1.1}]

FrequencyResponsePlot[ButterworthFilter[6, s], {s, 0, 2}]
```

Show[DiscreteSignal[$k^4\ 2^{-k}$, {k, 0, 15}, TopSize → 0.06`]]

Show[DiscreteSignal[$k^4\ 2^{-k}$, {k, 0, 15}]]

Show[DiscreteSignal[RightShift[5, k][Cos[$\frac{k}{5}$]], {k, 0, 25}]]

```
(s+Sqrt[s^2+a^2])/Sqrt[a^2+s^2]//AmplifyBy[-s+Sqrt[s^2+a^2]]

GenFac[5,3,a][k] /. ExpandGenFac

PfPair1[1+I,2,3+I,s] /. ExpandPfPair

Sqrt[(a + b I)^2 + (a - b I)^2] /. ExpandUnderSqrt

(a b Sin[b t] + a c Cos[2 b t] - 4 a) /. FactorOutOfSum[a]

A Cos[ω t] + B Sin[ω t] /. ToAmplitudePhaseForm

(Exp[(x-a)s]-Exp[-(x+a) s])/(s^2 (1+Exp[-2 a s]))
/. ToHyperbolicFunctions

ArcTan[-1,-3] /. ToPositiveAngle

Summation[(-1)^k Cos[Pi k t]/k, {k,Infinity}]
 /. TruncateSummation[10]

f[x] DiracDelta[ x + b] /. DiracDeltaRules

ReduceLagrangian[L[x[t], y[t], x'[t], y'[t]],
 {{x[t], x0}, {y[t], y0}, {x'[t], x0'}, {y'[t], y0'}}]

HurwitzTest[Sum[a[i] s^i, {i, 0, 4}], s]

JuryTest[z^3 - 3 z^2 + 2 z + b, z]

Vieta[2 + s + k (s + 1) + k s^2 - 7 s^3, s, σ]

Resolvent[s, {{1, 2}, {3, 4}}]
```

■ Differential and Integral Equations

```
LinearIDSolve[y''[t] + 2 y'[t] == -5 y[t] + Exp[-t] Sin[t],
{y[0]->0, y'[0]->1},y[t],t]

LinearIDSolve[y''[t] + 2 y'[t] == -5 y[t] + Exp[-t] Sin[t],
{y[0]->0, y'[0]->1},y,t]

LaplaceImageSystem[{ x''[t] - x[t] == 3 y'[t] + 3 Cos[t] - a t,
  y''[t] - x'[t] == t}, {x, y}, {X, Y}, t, s]

LinearIDSolve[y''[t] + 2 y'[t] == -5 y[t] + Exp[-t] Sin[t],
{y[0]->0, y'[0]->1},y[t],t, GoFor->Image[Y,s]]

LinearIDSolve[{x''[t] - 2 x'[t] == -3 y'[t],
y'[t] - y[t] == -2 x[t] + t},{x[0]->0,x'[0]->1},{x[t],y[t]},t]

LinearIDSolve[{x''[t] - 2 x'[t] == -3 y'[t],
y'[t] - y[t] == -2 x[t] + t},{x[0]->0,x'[0]->1},{x[t],y[t]},
t,GoFor->Image[{Y[2]},s]]

LinearIDSolve[y[t] == t^2 + Convolution[y[u],Sin[u],u][t],y[t],t]

LinearIDSolve[y[t] + y'[t]== t^2 + Convolution[y[u],Sin[u],
u][t],{y[0]->0},y[t],t]

LinearDOpSolve[(p^2 + 2 p + 5)[y] == Cos[2 t],p,y[t],t]

LinearDOpSolve[(p^2 + 2 p + 5)[y] == (p+3)[Exp[-t] Cos[2 t]],
p,y[t],t]

LinearDOpSolve[(p^2 + 2 p + 5)[y] == 0,{y[0]->3, y'[0]->-4},p,y[t],t]

LinearDOpSolve[(p^2 + 2 p + 5)[y] == (p+3)[Exp[-t] Cos[2 t]],
{y[0]->3, y'[0]->-4},p,y,t]

LinearDOpSolve[(p^2 + 2 p + 5)[y] == (p+3)[Exp[-t] Cos[2 t]],
{y[0]->3, y'[0]->-4},p,y[t],t,GoFor->Image[Y,s]]

MDOp1 = {{17 + 2 p + p^2, -5 + p, -3},
{-1 + 2 p, p, -p^2}, {3,4 - p, -p}};
MDOp2 = {{-1 + 3 p, p, 1},
{0, p^3, 0}, {p, 0, -5}};
LinearDOpSolve[MDOp1[{y1,y2,y3}] == MDOp2[{Sin[t], Exp[-t], t}],
p,{y1[t],y2[t],y3[t]},t]//Chop

LinearDOpSolve[MDOp1[{y1,y2,y3}] == MDOp2[{Sin[t], Exp[-t], t}],
{y1[0]->5},p,{y1[t],y2[t],y3[t]},t]//Chop

LinearDOpSolve[MDOp1[{y1,y2,y3}] == MDOp2[{Sin[t], Exp[-t], t}],
p,{y1[t],y2[t],y3[t]},t, GoFor->Image[{Y[1],Y[3]},s]]

ToDOpForm[a0 y[t] + a1 y'[t] + a2 y''[t] + a3 y'''[t]
== Cos[t],p,y,t]

SpecialDOpSolve[(p^4 + a^2/s^2)[y] == 0,p,y[x],x]

SpecialDOpSolve[(p^4 + a^2/s^2)[y] == x^3,p,y[x],x]

SpecialDOpSolve[(p^4 + a^2/s^2)[y] == x^3,p,y[x],x,
GoFor->OnlyParticular[]]

AnomalousSystemQ[{{p+p^2,1},{2+p^2,-p}}[{y1,y2}] ==
{10,t},p,{y1[t],y2[t]},t]

CompatibilityConditions[{{p+p^2,1},{2+p^2,-p}}[{y1,y2}] ==
{10,t},p,{y1[t],y2[t]},t]

VariableDSolve[t^2 y''[t] + t y'[t] + (t^2-n^2)y[t] == 0,t,Y,s]

StraightforwardExpansion[(p^2 + ω^2)[y]== ϵ y^3,
{y[0]->a Cos[φ],y'[0]-> -a ω Sin[φ]},p,y[t],t,2]
```

■ Nonlinear Problems

```
AutonomousOscillation1st[(1 - y^2) dy, y, dy, ω, ϵ, a, ψ, t]

AutonomousOscillation2nd[(1 - y^2) dy, y, dy, ω, ϵ, a, ψ, t]
```

Asymptpotics

```
AsymptoticSeries[1 / Sqrt[t] / (1 + Sqrt[t]) ^2, {t, 0, 4}]

AsymptoticSeries[LaplaceIntegral[1 / Sqrt[t] / (1 + Sqrt[t]) ^2, {t, 0}][s], {s, Infinity, 5}]

AsymptoticSeries[HankelIntegral[Exp[- a Sqrt[s]] / (s^2 + ω^2), {s, 0}][t], {t, Infinity, 5}]

AsymptoticSeries[
 LaplaceTypeIntegral1[c + t^3 (1 + t^2 + 3 t^4), Cos[t], {t, 0}][z], {z, Infinity, 6}]

AsymptoticSeries[
 LaplaceTypeIntegral1[b + (t - a) ^2 Cos[t], 1 / Sqrt[t - a] Exp[t], {t, a}][z],
 {z, Infinity, 3}]

AsymptoticSeries[Sin[t],{t,0,3}] /. RemoveLandauO

LaplaceMethodCoefficients[{p0, p1, p2}, {q0, q1, q2}, μ, λ]
```

Numerical Inversion of Laplace Transforms

With for example,

```
F = 1 / (s^2 + 2 s + 5);

f = NInverseLaplaceLaguerre1[F, s, t, 20, 0.1]

f = NInverseLaplaceLaguerre2[F, s, t, 32, 10, 0]

f = NInverseLaplaceFourier1[F, s, t, 0.1, {19, 11, 10}]

f = NInverseLaplaceFourier2[F, s, t, 256, 0, 12]

f = NInverseLaplaceWPS[F, s, t, {8, 10, 6}]

f = NInverseLaplaceGauss[F, s, t, 10, 1.5]

f = NInverseLaplaceGaverStehfest[F, s, t, 20]

Plot[f, {t, 0, 6}]
```

Finite Laplace and Fourier Transforms

```
FiniteFourierSineTransform[x (Pi - x), x, n]

InverseFiniteFourierSineTransform[1 / n^3, n, x]

FiniteFourierCosineTransform[x^2, x, n]

FFCosineConvolution[x^2, (x - L) ^2, L]

NFiniteInverseLaplaceTransform[(Exp[-s]-Exp[-2 s])/s,{s,0,3}]

FiniteLaplaceTransform[t/(2 Pi),{t,0,2 Pi},s]
```

Control and Linear Systems

- **Stability Test and Responses**

```
HurwitzTest[a + b s + 5 s^2 + c s^3, s]
```

```
StabilityQ[a + b s + 5 s^2 + c s^3, s]
```

```
ZStabilityTest[Product[(s+k/10),{k,10}],s]
```

```
JuryTest[z^3 - 3 z^2 + 2 z 1 + 1 / 2, z]
```

```
FrequencyResponse[s / (s^2 + 2 s + 5), s, a Cos[ω t + φ], t]
```

```
FrequencyResponse[s / (s^2 + 2 s + 5), s, a Sin[ω t + φ], t]
```

```
FrequencyResponsePlot[s / (s^2 + 0.2 s + 1), {s, 0, 3}]
```

```
BodePlot[s / (s^2 + 0.2 s + 1), {s, 0.5, 3}]
```

```
ContinuousArgPlot[Arg[ω I / (-ω^2 + 0.2 I ω + 1)], {ω, 0.8, 1.1}]
```

- **Filters**

```
ButterworthFilter[7, s]
```

```
ChebyshevFilter[7, 0.3, s]
```

- **Controller Configuration, Transfer Functions, System Response**

```
cc1 = StandardCC[-1, {1 / (s + 1)}, {(s - 1) / (s + 2)}, s]
```

```
cc2 = CascadeCC[-1, {2 / (s + 2)}, {s / (s^2 + s + 1), 1 / s}, s]
```

```
cc3 = FeedbackCC[-1, {s / (s^2 + s + 1)}, {s / (s + 2)}, s];
```

```
cc4 = SeriesSeriesCC[-1, {s / (s^2 + s + 1)}, {s / (s + 2)}, {s / (s + 1 / 2) ^2}, s]
```

```
cc5 = FeedForwardCC[-1, {s / (s^2 + s + 1)}, {s / (s + 2)}, {s / (s + 1 / 2) ^2}, s]
```

```
cc6 = PerturbationCascadeCC[-1, {s / (s^2 + s + 1)}, {s / (s + 2)}, {s / (s + 1 / 2) ^2}, s]
```

for example,

```
ClosedLoopTransferFunction[cc1]
```

```
ClosedLoopCharacteristicFunction[cc1]
```

Second Order Matrix Differential System:

```
A0 = {{1, 1}, {0, -1}}; A1 = {{1, -1}, {0, -2}}; A2 = {{0, 1}, {2, -1}};
B0 = {{1, 0}, {0, 2}}; B1 = {{3, 1}, {4, -1}};
H = {{1}, {1}}
```

```
sys = MatrixDifferentialSystem[{A2, A1, A0}, {B0, B1}];
TransferMatrices[sys, H, s]
```

```
K0 = {{k, 0 Bm = {{0, 0}, {1, 0}, {0, 1}};}, {0, -1}}; K1 = IdentityMatrix[2]
```

```
cc7 = FullFeedbackCC[sys, {K0, K1}, s];
ClosedLoopTransferMatrix[cc7, H]
```

```
ClosedLoopCharacteristicFunction[cc7]
```

```
DynamicSystemResponse[sys, H, {K0, K1}, {0, 0}, {0, 0}, {UnitStep[t]}, t, s]
```

State Control Systems:

```
Am = {{0, 1, 0}, {0, -4, 3}, {-1, -1, -2}};
Bm = {{0, 0}, {1, 0}, {0, 1}};
Cm = {{1, 0, 0}, {0, 0, 1}};
Dm = NullMatrix[2]; Hm = NullMatrix[2, 1];
Em = {{0}, {2}, {0}}; σ
sys = StateControlSystem[Am, Bm, Cm, Dm];
TransferMatrices[sys, Em, Hm, s]

DynamicSystemResponse[sys, Em, Hm][{1, t}, {DiracDelta[t]}, t, s]

DynamicSystemResponse[sys, Em, Hm, {1, 2, 3}][{1, t}, {DiracDelta[t]}, t, s]

A = {{0, 1}, {-2, 3}};
StateTransitionMatrix[A, t]

Am = {{0, 1, 0}, {1, 0, 0}, {1, 1, -1}}; Cm = {{1, 0, 0}, {1, -1, 2}};
ObservabilityQ[Am, Cm]

Bm = {{1}, {0}, {0}};
ControllabilityQ[Am, Bm]

Cm1 = {{1, 2, 3}};
OutputControllabilityQ[Am, Bm, Cm1]
```

- **Discrete System Control**

```
Am = {{0, 1}, {-5, -2}}; Bm = {{0}, {1}}; Cm = IdentityMatrix[2]; Em = NullMatrix[2, 1];
dsys = ToDiscreteStateControlSystem[Am, Bm, Cm, Em, 0.5]

DiscreteDynamicSystemResponse[dsys[[1]], dsys[[2]], dsys[[3]]][
  {UnitStep[k]}, {0}, k, z] // Chop
```

- **Miscellaneous:**

```
EvansForm[1 + k s + (1 - k) s^2 + s^3, s, k]

Vieta[1 + k s + (1 - k) s^2 + s^3 - 2 k s^4, s, σ]
```

Discrete Signals, Z-Transforms and Inverse Z-Transforms

- **Discrete Signals**

```
Show[DiscreteSignal[k^4 2^-k, {k, 0, 15}]]

Show[DiscreteSignal[RightShift[5, k][Cos[k/5]], {k, 0, 25}]]

Show[DiscreteSignal[LeftShift[5, k][Cos[k/5]], {k, 0, 25}]]
```

- **Z- Transforms**

```
ZTransform[k^2,k,z]

ZTransform[k(k-1)Exp[-a k] + 5 k^3 -b k(k-1)a^-k,k,z]

ZTransform[Product[k-j,{j,0,7}]//Unevaluated,k,z]
```

```
ZTransform[Binomial[5+k,5]//Unevaluated,k,z]

ZTransform[Sum[a^i,{i,0,k}]//Unevaluated,k,z]

ZTransform[Sum[a^i,{i,0,k-1}]//Unevaluated,k,z]

ZTransform[RightShiftRule[5,k][k^2],k,z]

ZTransform[k^2//LeftShiftRule[5,k],k,z]

ZTransform[GenFac[5,3,a][k],k,z]
```

- **Inverse Z-Transforms**

```
InverseZTransform[(z - a) / (z + b), z, k]

InverseZTransform[5 z/(z-a)^3 + b/(z+3) + z/(z-1),z,k]

F = (z^2 - z) / ((z^2 + 0.1 z + 0.95)^2 (z^2 + 0.3 z + 0.8));
f = InverseZTransformInitialData[F, z, 20]

Show[DiscreteSignal[f], PlotRange → All]
```

- **Discrete Convolutions**

```
DiscreteConvolution[a^j,j^2,j][k] /. ComputeDiscreteConvolution
```

Advanced Z-Transforms and their Inverses

- **Advanced Z-Transforms**

```
AdvancedZTransform[t^2 Cos[a t],t,T,τ ,z]

AdvancedZTransform[Sin[t]//RightShiftRule[4,t],t,T,τ,z]

AdvancedZTransform[Sin[t]//LeftShiftRule[4,t],t,T,τ,z]

AdvancedZTransform[t^2//RightShiftRule[1.5,t],t,2,0.7,z]

AdvancedZTransform[Cos[t]//IntegrationTo[t],t,T,τ,z]

AdvancedZTransform[(t^5)'',t,T,τ,z]
```

- **Inverse Advanced Z-Transform**

```
F = AdvancedZTransform[t^3,t,T,τ ,z];
rf = InverseAdvancedZTransform[F,z,f,k,T,τ ] /. ExpandGenFac

f= EvaluateToFunction[rf,1.5]

Plot[f[t], {t, 0, 10}]
```

- **Laplace Image Function to Z-Image Function**

```
LaplaceToZImage[(s+1)/((s+2)(s^2 + 2 s + 17)), s,T,τ ,z]
```

- **Discrete Compositions**

```
DiscreteComposition[Exp[-x],Sin[x],x,T][t,τ ]/.
ComputeDiscreteComposition
```

Difference Equations

Discrete Difference Equations

```
DiscreteSystemSolve[y[k+2] + 2 y[k+1] + 5 y[k] == k^2,
{y[0]→0,y[1]→1},y[k],k]
```

```
DiscreteSystemSolve[y[k+2] + 2 y[k+1] + 5 y[k] == k^2,
y[k],k,GoFor->Image[Y,z]]
```

```
eqn = y[k] + DiscreteConvolution[y[i],2^i,i][k] == 3^k;
DiscreteSystemSolve[eqn,y[k],k]
```

```
sys = {a[k+1] == 3 a[k] + 4 c[k],b[k+1] == 3 b[k] + 4 d[k],
c[k+1] == a[k] + 3 c[k],d[k+1] == b[k] + 3 d[k]};
iv = {a[0]->1, b[0]->0,c[0]->0,d[0]->1};
DiscreteSystemSolve[sys,iv,{a[k],b[k],c[k],d[k]},k]
```

```
ToDiscreteDOpForm[y[k + 2] + 2 y[k + 1] + 5 y[k] == k^2,
 T, y, k]
```

```
DiscreteSystemSolve[sys,iv,{a[k],b[k],c[k],d[k]},k,
GoFor->Image[{Y[1],Y[4]},k]]
```

```
DiscreteDOpSolve[(T^2 - 6 T + 8)[y] == (T + 1)[k^2],T,y[k],k]
```

```
DiscreteDOpSolve[(T^2 - 6 T + 8)[y] == (T + 1)[k^2],
{y[0]->1,y[1]->-5},T,y[k],k]
```

```
DiscreteDOpSolve[(T^2 - 6 T + 8)[y] == (T + 1)[k^2],T,y[k],k,
GoFor->Image[Y,z]]
```

```
dop1 = {{2 T - 1, T - 2},{T + 3,T + 4}}; dop2 = {{T,1},{T^2,0}};
DiscreteDOpImSolve[dop1[{x,y}] == dop2[{k^2, 2^k}],
{x[0]->0,y[0]->0},{X,Y},T,k,z]
```

```
ToShiftedForm[(T^2 - 6 T + 8)[y] == k^2, T, y[k], k]
```

Continuous Difference Equations

```
deqn = y[t - 2 T] == 4 y[t - T] - 3 y[t] + t;
BackwardDifferenceEquationSolve[deqn,y[t],t,T,τ,k]
```

```
BackwardDifferenceEquationSolve[deqn,y[t],t,T,τ,k,
GoFor->Image[Y,z]]
```

```
deqn = y[t + 2 T] == - 2 y[t + T] - y[t] == Sin[t];
ForwardDifferenceEquationSolve[deqn, {y[τ ]->τ ,y[τ +T]->T-τ },
y[t],t,T,τ,k]
```

```
ForwardDifferenceEquationSolve[deqn, {y[τ ]->τ ,y[τ +T]->T-τ },
y[t],t,T,τ ,k, GoFor->Image[Y,z]]
```

Definitions and Rules

Definitions

```
DefReal[n]; DefInteger[n]; DefIntegerPositive[n];
DefIntegerNonNegative[n]; DefEven[n]; DefOdd[n];
```

```
DefPositive[x]; DefNegative[x]; DefNonNegative[x];
```

```
DefGreater[a, b]; DefGreaterEqual[a, b]; DefLess[a, b]; DefLessEqual[a, b]
```

Rules

```
GenFac[5,3,a][k] /. ExpandGenFac
```

```
PfPair1[1+I,2,3+I,s] /. ExpandPfPair

a(Cos[φ ] Cos[ω t] - Sin[φ ] Sin[ω t]) /. AdditionTheorems

A Cos[ω t] + B Sin[ω t] /. ToAmplitudePhaseForm

ArcTan[-1,-3] /. ToPositiveAngle

Exp[c(a + b I) + c(a - b I)] /. SimplifyExp

Sqrt[a + b I] /. CartesianSqrt

(Cos[n(ω t - π )] - Sin[n(ω t - π )]) /. CosSinWithIntegerPi

Exp[(Sqrt[a^2+b^2] +Sqrt[a^4+b^4] I)(x+y)] /. EulerRule

Sqrt[(a + b I)^2 + (a - b I)^2] /. ExpandUnderSqrt

(a b Sin[b t] + a c Cos[2 b t] - 4 a) /. FactorOutOfSum[a]

InverseLaplaceTransform[Exp[-a s + b]/Sqrt[s^2+a^2],
s,t] /. ImageRules

InverseLaplaceTransform[Hold[(s/(s^2+a^2))/. s->s+c],
s,t] /. ImageRules

InverseLaplaceTransform[Hold[1/Sqrt[s^2+1]/s],
s,t] /. ImageRules

Convolution[Exp[-x],Sin[x],x][t] /. ComputeConvolution

Convolution[Exp[-x],Sin[x],x][t] /. EvaluateConvolution

DiscreteComposition[Exp[-x],Sin[x],x,T][t,τ]
/. ComputeDiscreteComposition

DiscreteConvolution[a^j,j^2,j][k]
/. EvaluateDiscreteConvolution

AsymptoticSeries[Sin[t],{t,0,3}] /. RemoveLandauO

(Exp[(x-a)s]-Exp[-(x+a) s])/(s^2 (1+Exp[-2 a s]))
/. ToHyperbolicFunctions

Summation[(-1)^k Cos[Pi k t]/k, {k,Infinity}]
 /. TruncateSummation[10]

f[x] DiracDelta[x + b] /. DiracDeltaRules
```